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On the scalar particle creation by electromagnetic fields in Robertson–Walker spacetime

Kenan Sogut [*], Ali Havare

Mersin University, Department of Physics, 33343, Mersin, Turkey

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Abstract

In the present paper, we obtained the scalar particle creation number density by using the Klein–Gordon equation coupled to the electromagnetic fields in the Robertson–Walker spacetime with the help of the Bogoliubov transformation method. We analyzed the resulting expression for the effect of a time-dependent electric field and a constant magnetic field on the particle production rate and found that the strong time-dependent electric field amplifies the particle creation and the magnetic field reduces the rate, in accordance with the previous findings.

1. Introduction

In quantum electrodynamics (QED), the problem concerning the creation of particles out of the unstable Minkowski vacuum in the presence of strong electric fields is called "Schwinger effect" [1]. This problem has been worked in detail by [2–4].

Studying the quantum field theory (QFT) in curved spacetimes is a way of understanding the seemingly irrelevant concepts of the quantum mechanics and gravity. The creation of elementary particles out of the unstable vacuum by gravitational fields is a well known problem by those who attempt to employ QFT in the study of cosmology. Instability of the vacuum is also caused

* Corresponding author.
E-mail address: kenansogut@gmail.com (K. Sogut).

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by the contraction or expansion of the spacetime [5–8]. Since the unstable vacuum state in the early stage alters the late stage of the universe, the discovery of the 2.7 K cosmic microwave background radiation has revealed the presence of strong electromagnetic fields in the early times of the universe [9]. Therefore, the contributions of the electromagnetic fields to the gravitational fields must be evaluated in the study of QFT in curved spacetimes.

The particle creation problem in the case of interacting electromagnetic and gravitational fields has been studied [9–13] less often, compared to the cases containing the pure gravitational or electromagnetic fields. In the present study, we compute the number density of the particles created in the presence of both electromagnetic and gravitational fields by using vacuum mode solutions. We consider a time-dependent electric field and a homogeneous magnetic field generated by the following 4-vector electromagnetic potential

$$A_\nu = yB_0\delta_\nu^1 + E_0\sqrt{\Gamma + \Lambda t}\,\delta_\nu^3 \tag{1}$$

where $\nu = 0, 1, 2, 3$, and B_0, E_0 are constants.

The time-dependent expanding universe models have an intrinsic structure of creating particles from the vacuum. In our study we consider the Robertson–Walker (RW) spacetime representing the early universe [14]

$$ds^2 = \frac{1}{a^2(t)}dt^2 - a^2(t)(dx^2 + dy^2 + dz^2) \tag{2}$$

where the scale factor $a(t) = \sqrt{\Gamma + \Lambda t}$ and Γ and Λ are constants. By making two successive changes of variables as $d\eta = \frac{dt}{a(t)}$ and $d\tau = \frac{d\eta}{a(\eta)}$, the metric takes the following form

$$ds^2 = a^2(\tau)[d\tau^2 - (dx^2 + dy^2 + dz^2)] \tag{3}$$

In order to study the quantum effects in curved spacetime, we have to identify the positive and negative-frequency vacuum states for the asymptotic regions of the universe. There are several approaches widely used for this purpose, such as the Feynman path-integral method [15], Hamilton diagonalization method [16], adiabatic method [17] and semi-classical approach based on the solutions of the relativistic Hamilton–Jacobi equation [18,19], which we will use for our calculations. The particle creation number density will be calculated via Bogoliubov transformation by using "in" and "out" vacuum solutions [20].

The structure of the article will be as follows: In Section 2 we solve the relativistic Hamilton–Jacobi equation and, by using the asymptotic behavior of these mode solutions, we identify the positive and negative-frequency states. In Section 3 we obtain exact solutions of the Klein–Gordon equation in the presence of electromagnetic fields in RW spacetime. Based on the solutions given in Section 2, we identify the "in" and "out" states. We then compute the number density of created particles via Bogoliubov coefficients in Section 4, by using the relation between these solutions. Finally, we discuss the results for the particular cases of the electromagnetic fields in Section 5. Throughout the paper, Greek indices run from $0, \ldots, 3$ and the Heaviside units $c = \hbar = 1$ are used.

2. Solution of the Hamilton–Jacobi equation

Because of the optional time dependence of the external electromagnetic field, it is difficult to get the exact "in" and "out" vacuum solutions. Therefore, we obtain the solutions of the relativistic Hamilton–Jacobi equation in order to discuss the asymptotic behavior of the solutions of the Klein–Gordon equation.

The relativistic Hamilton–Jacobi equation for the action S is given by [21]

$$g^{\epsilon\theta}\left[\frac{\partial S}{\partial x^{\epsilon}} - eA_{\epsilon}\right]\left[\frac{\partial S}{\partial x^{\theta}} - eA_{\theta}\right] + m^2 = 0 \tag{4}$$

where $A_{\epsilon,\theta}$ is the 4-vector electromagnetic potential, m is the mass of the particle.

Taking into account the dependence of the A_{ϵ} to the space coordinate y and time coordinate τ and the dependence of the line element given by (3) to the τ, the solution of the Hamilton–Jacobi equation can be separated in the following form:

$$S(\tau, x, y, z) = F(\tau) + T(y) + (xk_x + zk_z). \tag{5}$$

Here k_x and k_y can be viewed as the conserved momenta that exist given the symmetries chosen for the electromagnetic field given in (1) and the line element given in (3). Substitution of (5) into (4) yields

$$\dot{F}^2 - \left[(k_x - eB_0 y)^2 + \acute{T}^2 + (k_z - eE_0 a(\tau))^2\right] + m^2 a^2(\tau) = 0 \tag{6}$$

where dot and acute denote derivatives with respect to τ and y, respectively. We obtain two first order differential equations as follows:

$$\left[\dot{F}^2 - (k_z - eE_0 a(\tau))^2\right] + m^2 a^2(\tau) = r^2 \tag{7}$$

and

$$\acute{T}^2 + (k_x - eB_0 y)^2 = r^2 \tag{8}$$

where r is the constant of separation.

Since the time-dependent external fields and strong gravitational fields cause particle creation, the dynamics involving spatial coordinates effect the solutions by a constant, and quasi-classical behavior of the solution of the Hamilton–Jacobi equation for metric (3) and external electromagnetic field (1) is obtained as

$$\begin{aligned}\Phi(\tau, x, y, z) = e^{iS} &\rightarrow C(x, y, z)\sqrt{e^2 E_0^2 - m^2} \\ &\times \left\{\sqrt{e^{2a_0\tau} + be^{a_0\tau} + c} + \frac{b}{2} Sinh^{-1}\left(\frac{2e^{a_0\tau} + b}{\sqrt{4c - b^2}}\right) - \sqrt{c}\, Sinh^{-1}\left(\frac{be^{a_0\tau} + 2c}{e^{a_0\tau}\sqrt{4c - b^2}}\right)\right\}\end{aligned} \tag{9}$$

where $a_0 = \frac{\Lambda}{2}$, $b = \frac{-2k_z e E_0}{a_0(e^2 E_0^2 - m^2)}$ and $c = \frac{(2n+1)eB_0 + k_z^2}{a_0(e^2 E_0^2 - m^2)}$.

The asymptotic behavior of this solution is given by

$$\Phi_{(\tau \rightarrow -\infty)} \rightarrow C(x, y, z) e^{\pm i \frac{\sqrt{(2n+1)eB_0 + k_z^2}}{a_0}\tau} \tag{10}$$

and

$$\Phi_{(\tau \rightarrow +\infty)} \rightarrow C(x, y, z) e^{\pm i \sqrt{e^2 E_0^2 - m^2} e^{a_0\tau}} \tag{11}$$

where the upper and lower signs in the exponential functions represent the negative and positive-frequency states, respectively. The “in” and “out” solutions of the Klein–Gordon equation can be determined with the aid of this analysis.

3. Solution of the Klein–Gordon equation

The Lagrangian density of a scalar field $\Psi(\tau, x, y, z)$ interacting with a curved spacetime geometry is given by [20]

$$\mathcal{L} = \frac{1}{2}\sqrt{-g}\left[g^{\varepsilon\phi}(x)\Psi_{,\varepsilon}(x)\Psi_{,\phi}(x) - \left(m^2 + \xi R(x)\right)\Psi^2\right] \tag{12}$$

where comma $(,)$ represents the ordinary partial derivatives (∂_ε) and m is the mass of the field quanta, $R(x)$ is the Ricci scalar, ξ is a numerical factor. This form of the Lagrangian is invariant under the *global* gauge transformations.

The Lagrangian density of the scalar field coupled to the electromagnetic fields in curved spacetime requires the *local* gauge invariance [22]. The general form of the *local* gauge transformations is given as

$$\Psi(x) \to \overline{\Psi}(x) = e^{i\zeta(x)}\Psi(x) \tag{13}$$

where $\zeta(x)$ is an arbitrary function of the spacetime coordinates. With this definition the partial derivative of the scalar field in the Lagrangian becomes

$$\Psi_{,\varepsilon}(x) \to \overline{\Psi}_{,\varepsilon}(x) = e^{i\zeta(x)}(\Psi_{,\varepsilon}(x) + i\zeta_{,\varepsilon}\Psi) \tag{14}$$

Hence, to obtain the *local* gauge invariance of the Lagrangian density the ordinary partial derivatives are replaced by the gauge covariant derivatives D_α,

$$\Psi_{,\varepsilon} \to D_\varepsilon\Psi \equiv \Psi_{,\varepsilon} + iA_\varepsilon\Psi \tag{15}$$

and the following transformation law is postulated for the gauge field A_α,

$$A_\varepsilon \to \overline{A}_\varepsilon \equiv A_\varepsilon - \zeta_{,\varepsilon} \tag{16}$$

Then, it follows that the Lagrangian density of a scalar field interacting with the electromagnetic fields in curved spacetime

$$\mathcal{L} = \frac{1}{2}\sqrt{-g}\left[g^{\varepsilon\phi}(x)(D_\varepsilon\Psi)(D_\phi\Psi) - \left(m^2 + \xi R(x)\right)\Psi^2\right] \tag{17}$$

is invariant under the *local* gauge transformations.

The Lagrangian density (17) leads to the Klein–Gordon equation for scalar spin-0 particles coupled to the electromagnetic fields in curved spacetime [21]:

$$\left[g^{\varepsilon\phi}(\nabla_\varepsilon - ieA_\varepsilon)(\nabla_\phi - ieA_\phi) - (m^2 + \xi R)\right]\Psi(\tau, x, y, z) = 0 \tag{18}$$

where ∇_ε is the covariant derivative, $A_{\varepsilon,\phi}$ is the 4-vector electromagnetic potential, m is the mass of the scalar particle and ξ is a dimensionless coupling constant which is zero for the minimally coupled case.

Since the 4-vector potential (1) and the line element (3) depend on space variable y and time variable τ, we introduce solution of the Klein–Gordon equation (18) in the following form

$$\Psi(\tau, x, y, z) = e^{i(xk_x + zk_z)}V(\tau)G(y) \tag{19}$$

Then, substituting (1) and (3) into the Klein–Gordon equation we obtain

$$\left[\widehat{P}(\tau) + \widehat{Q}(y)\right]V(\tau)G(y) = 0 \tag{20}$$

where the following definitions are made

$$\widehat{P} = \left[\partial_\tau^2 + (k_z - eE_0 a_0 e^{a_0\tau})^2 - m^2 a^2(\tau)\right] \tag{21}$$

$$\widehat{Q} = \left[(k_x - eB_0 y)^2 - \partial_y^2\right] \tag{22}$$

According to these definitions equation (20) can be separated into two independent equations as follows

$$(\widehat{P} + s)V(\tau) = 0 \tag{23}$$

$$(\widehat{Q} - s)G(y) = 0 \tag{24}$$

where s is a constant obtained from the separation.

By defining $\gamma = \sqrt{e^2E_0^2 - m^2}$, $\rho = 2i\gamma e^{a_0\tau}$ and writing the wave function associated the conformal time as $V(\tau) = e^{-\frac{a_0\tau}{2}} v(\tau)$, we obtain from Eq. (23)

$$\left[\frac{d^2}{d\rho^2} + \frac{1}{\rho^2}\left(\frac{1}{4} + \frac{s + k_z^2}{a_0^2}\right) + \frac{1}{\rho}\frac{ik_z eE_0}{a_0\gamma} - \frac{1}{4}\right] v(\rho) = 0 \tag{25}$$

This is the Whittaker equation and its solutions are given by [23]

$$v(\rho) = \left[AW_{\lambda,\mu}(\rho) + BM_{\lambda,\mu}(\rho)\right] \tag{26}$$

where $\mu = \pm i\frac{\sqrt{s+k_z^2}}{a_0}$, $\lambda = \frac{ik_z}{a_0\sqrt{1-(\frac{m}{eE_0})^2}}$ and n is integer and A, B are arbitrary constants.

We now look for the solution of Eq. (24), which yields the differential equation of the harmonic oscillator for the dimensionless variable $u = \sqrt{eB_0}\left(y - \frac{k_x}{eB_0}\right)$ and has the solution

$$G(u) = e^{-\frac{u^2}{2}} H_n(u) \tag{27}$$

where $H_n(u)$ are the Hermite polynomials, $s = (2n+1)eB_0$ and n is an integer.

Hence, the exact solution of the Klein–Gordon equation is given by

$$\Psi = e^{i(xk_x + zk_z)} e^{-\frac{u^2 + a_0\tau}{2}} H_n(u)\left[AW_{\lambda,\mu}(2i\gamma e^{a_0\tau}) + BM_{\lambda,\mu}(2i\gamma e^{a_0\tau})\right] \tag{28}$$

4. Particle creation

The particle creation number density will be calculated by identifying the Bogoliubov coefficients based on the exact mode solutions. In the QFT, the orthogonal solutions of the field equation, Ψ, are written in terms of the mode solutions as follows [20]

$$\Psi = \sum_n (a_n\tau_n + a_n^\dagger\tau_n^*) = \sum_k (b_k\Upsilon_k + b_k^\dagger\Upsilon_k^*) \tag{29}$$

where τ and Υ are mode solutions and satisfy the relations $(\tau_i, \tau_j) = \delta_{ij}$, $(\tau_i^*, \tau_j^*) = \delta_{ij}$, $(\tau_i, \tau_j^*) = 0$ and $(\Upsilon_i, \Upsilon_j) = \delta_{ij}$, $(\Upsilon_i^*, \Upsilon_j^*) = \delta_{ij}$, $(\Upsilon_i, \Upsilon_j^*) = 0$.

Since the mode-solutions are complete, they can be expanded in terms of the each other. $a_n^\dagger, b_k^\dagger$ and a_n, b_k are creation and annihilation operators, respectively and are connected by the following expressions

$$a_n = \sum_k (\alpha_{kn} b_k + \beta^*_{kn} b^\dagger_k) \tag{30}$$

$$b_k = \sum_n (\alpha^*_{kn} a_n - \beta^*_{kn} a^\dagger_n) \tag{31}$$

α_{kn} and β_{kn} are defined as the Bogoliubov coefficients which are evaluated by $\alpha_{ij} = (\Upsilon_i, \tau_j)$, $\beta_{ij} = -(\Upsilon_i, \tau^*_j)$. The Bogoliubov coefficients satisfy the below relations:

$$\sum_i (\alpha_{ni}\alpha^*_{ki} - \beta_{ni}\beta^*_{ki}) = \delta_{nk} \tag{32}$$

$$\sum_i (\alpha_{ni}\beta_{ki} - \beta_{ni}\alpha_{ki}) = 0. \tag{33}$$

Two different states of vacua in the Fock space, $|0_a\rangle$ and $|0_b\rangle$, are related to each particle notion in (29), and they can be defined for all n and k as follows:

$$|0_a\rangle : a_n|0_a\rangle = 0 \tag{34}$$

$$|0_b\rangle : b_k|0_b\rangle = 0 \tag{35}$$

If $|0_b\rangle$ is selected to be a natural vacuum, then $|0_a\rangle$ is considered as a many-particle state. Then the number of Υ_n-mode particles in the state of $|0_a\rangle$ is given by

$$\langle 0_a|b^\dagger_k b_k|0_a\rangle = \sum_n |\beta_{kn}|^2 \tag{36}$$

If the $\tau_n(x)$ modes are positive frequency modes and the $\Upsilon_n(x)$ modes are linear combination of them, then $\beta_{jk} = 0$. In that case, $b_k|0_b\rangle = 0$ and $a_k|0_a\rangle = 0$. Thus τ_j and Υ_k modes share a common vacuum state. If $\beta_{jk} \neq 0$, then Υ_k contain a mixture of positive-τ_k and negative-τ^*_k frequency modes, namely the Fock space based on $a_k|0_a\rangle$ is associated with the Fock space based on $b_k|0_b\rangle$.

Time-dependent components of the wave-function (28), the Whittaker functions, will cause the particle creation. Hence, by taking into consideration the quasi-classical solutions obtained in Section 2, we can define the positive- and negative-frequency solutions in order to find the Bogoliubov coefficients.

The asymptotic behavior of $W_{\lambda,\mu}(\rho)$ for $\rho \to \infty$ is [23]

$$W_{\lambda,\mu}(\rho) \to e^{-\frac{\rho}{2}} \rho^\lambda \tag{37}$$

Then as $\rho \to \infty$ ($\tau \to \infty$) the positive and negative frequency modes are

$$f^+_\infty = A^+_\infty W_{\lambda,\mu}(\rho) \tag{38}$$

$$f^-_\infty = [A^+_\infty W_{\lambda,\mu}(\rho)]^* = A^-_\infty W_{-\lambda,\mu}(-\rho) \tag{39}$$

Therefore, the asymptotic behavior of the solution of (25) has the following form for $\rho \to \infty$

$$f(\rho) = A^+_\infty W_{\lambda,\mu}(\rho) + A^-_\infty W_{-\lambda,\mu}(-\rho) \tag{40}$$

which has the similar behavior of (11).

Analogously, the asymptotic behavior of $M_{\lambda,\mu}(\rho)$ as $\rho \to 0$ ($\tau \to -\infty$) is [23]

$$M_{\lambda,\mu}(\rho) \to \rho^{\mu+\frac{1}{2}} \tag{41}$$

and positive and negative frequency modes as $\rho \to 0$ are

$$f_0^+ = B_0^+ M_{\lambda,\mu}(\rho) \tag{42}$$

$$f_0^- = [B_0^+ M_{\lambda,\mu}(\rho)]^* = B_0^-(-1)^{-\mu+\frac{1}{2}} M_{\lambda,-\mu}(\rho) \tag{43}$$

where the coefficients B are reel arbitrary constants. Then the asymptotic behavior of the solution of (25) has the following form for $\rho \to 0$

$$f(\rho) = B_0^+ M_{\lambda,\mu}(\rho) + B_0^-(-1)^{-\mu+\frac{1}{2}} M_{\lambda,-\mu}(\rho) \tag{44}$$

which has the similar behavior of (10).

The positive-frequency mode at $\rho \to \infty$ can be written as a linear combination of the positive and negative frequency modes at $\rho \to 0$ in the form:

$$f_\infty^+(\rho) = \alpha f_0^+(\rho) + \beta f_0^-(\rho) \tag{45}$$

By using the following relation of the Whittaker functions [23],

$$W_{\lambda,\mu}(\rho) = \frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-\lambda)} M_{\lambda,\mu}(\rho) + \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}+\mu-\lambda)} M_{\lambda,-\mu}(\rho) \tag{46}$$

we find α and β coefficients as follows:

$$\alpha = \frac{A_\infty^+}{B_0^+} \frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-\lambda)} \tag{47}$$

and

$$\beta = \frac{A_\infty^+}{B_0^-} \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}+\mu-\lambda)} (e^{i\pi})^{\mu-\frac{1}{2}} \tag{48}$$

Then, we obtain the following relation for the Bogoliubov coefficients

$$\frac{|\alpha|^2}{|\beta|^2} = e^{2\pi\tilde{\mu}} \frac{|\Gamma(\frac{1}{2}+\mu-\lambda)|^2}{|\Gamma(\frac{1}{2}-\mu-\lambda)|^2} \tag{49}$$

where $\mu = i\tilde{\mu}$ and $\tilde{\mu} = \frac{\sqrt{k_z^2+(2n+1)eB_0}}{a_0}$.

Considering the following relation for the Gamma functions [23]

$$|\Gamma(\frac{1}{2}+iz)|^2 = \frac{\pi}{\cosh \pi z} \tag{50}$$

Eq. (43) reduces to

$$\frac{|\alpha|^2}{|\beta|^2} = \frac{e^{2\pi\tilde{\mu}} \cosh \pi\tilde{\sigma}}{\cosh \pi\sigma} \tag{51}$$

where the following definitions are made

$$\sigma = \left[\tilde{\mu} - \frac{k_z}{a_0\sqrt{1-(\frac{m}{eE_0})^2}} \right] \text{ and } \tilde{\sigma} = -\left[\tilde{\mu} + \frac{k_z}{a_0\sqrt{1-(\frac{m}{eE_0})^2}} \right].$$

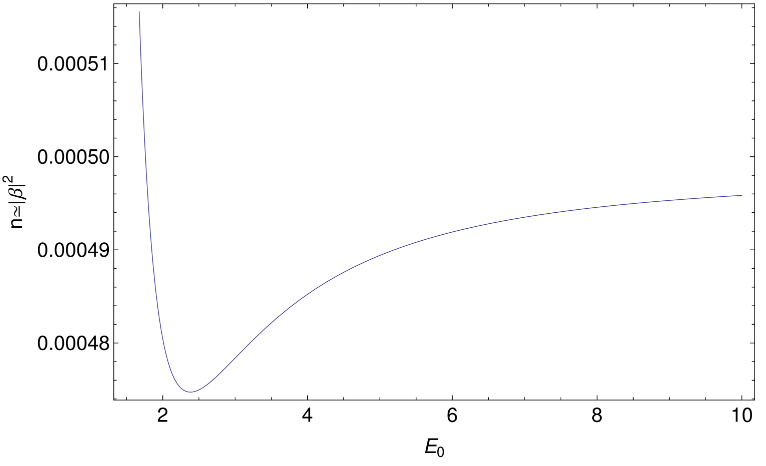

Fig. 1. Number density of created scalar particles against the strength of the electric field. e, k_z, B_0 are set to 1.

By using the normalization condition of the wave function (due to the Bose–Einstein statistics) $|\alpha|^2 - |\beta|^2 = 1$ and Eq. (51), the number density of the created scalar particles can be computed as follow

$$n \simeq |\beta|^2 = \frac{\cosh \pi\sigma}{e^{2\pi\tilde{\mu}} \cosh \pi\tilde{\sigma} - \cosh \pi\sigma} \tag{52}$$

5. Conclusion

We have discussed the particle creation mechanism by electromagnetic fields in the Robertson–Walker spacetime. For calculating the number density of created scalar particles via the Bogoliubov coefficients, we identified the “in” and “out” vacuum solutions by solving the Hamilton–Jacobi equation.

The number density of the created particles in Eq. (52) is not a thermal distribution. It takes the following thermal distribution form as the strength of the electromagnetic field becomes stronger

$$n \simeq |\beta|^2 \simeq e^{-\frac{2\pi k_z}{a_0}\left[\sqrt{1+\frac{(2n+1)eB_0}{k_z^2}}+\sqrt{1-(\frac{m}{eE_0})^2}\right]} \tag{53}$$

The temperature depends on the strength of the electric and magnetic fields. This expression shows that the strength of the electric field is restricted in the region $-\frac{m}{e} < E_0 < \frac{m}{e}$. This limitation is also required in order the solutions given by equations (11) and (26) to be finite.

In Fig. 1 the dependence of the number density of the created scalar particles versus to the strength of electric field is depicted. As it is seen from the figure there is a critical value for the strength of the electric field at which particle creation rate starts to increase. This value is obtained as to be $E_0 = |2.381|$ from the figure.

Individual contributions of the electric and magnetic fields to the particle creation number density can also be useful. For the vanishing magnetic fields, it follows $\tilde{\mu} = \frac{k_z}{a_0}$, and evaluation of Eq. (52) shows the number of the created particles increases due to the stronger values of the electric field. In the case of the vanishing electric fields, the stronger magnetic fields cause to reduction in the number of the created particles and Eq. (52) becomes to a thermal Bose–Einstein distribution as follow

$$n \simeq |\beta|^2 = \frac{1}{e^{\frac{2\pi\sqrt{k_z^2+(2n+1)eB_0}}{a_0}} - 1} \tag{54}$$

which is still in thermal distribution form in the case of pure gravitational fields as

$$n \simeq |\beta|^2 = \frac{1}{e^{\frac{2\pi k_z}{a_0}} - 1} \tag{55}$$

These results are in agreement with those presented by [8–13].

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